JOURNAL OF MATERIALS SCIENCE **38** (2003) 1123–1134

Indications that the yield point at constant strain rate and the inception of tertiary creep are manifestations of the same failure criterion using the universal viscoelastic model

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In a preceding publication this author introduced a new universal viscoelastic model to describe a definitive relationship between constant strain rate, creep and stress relaxation analysis for viscoelastic polymeric compounds. Since creep failure criterion for this model had not been addressed in detail in previous publications, selected creep failure criterion for this model were addressed in this study.

The first manifestation of the yield stress failure criterion as applied to creep was elucidated at the intersection of the yield stress relaxation curve and the creep stress vs time curve. A second way to apply yield point failure criterion to creep failure was through the identification of a specific creep time associated with the limiting strain to yield, ε_∞. The creep strain at ε_∞ occurs at the very end of the straight line portion of secondary creep and is also the strain at which tertiary creep appears to be initiated, $\varepsilon_{\text{itc}} = \varepsilon_\infty$.

As the strain increases from the inception of tertiary creep, ε_{itc}, eventually a strain is reached where a calculation option using this model would require a step back in time to go to the next differential element of strain. Since going back in time is currently impossible, only a huge jump in strain obtained by another calculation option for the next element of time would be realistic. Since this critical creep strain, ε_{CC}, is slightly greater than the inception of tertiary creep, if failure did not occur at the inception of tertiary creep then it would almost surely be expected to fail catastrophically at this condition.

The near equivalency of the critical creep strain criterion and the yield strain criterion was found to be much more probable the lower the value of efficiency of yield energy dissipation such that $0 < n \ll .4$. © *2003 Kluwer Academic Publishers*

1. Introduction

The extended use of finite element analysis with polymeric compounds [1] and composites [2] in recent years has generated a need for a simple analysis approach that relates creep, stress relaxation and constant strain rate measurements all in one simple model. One such unifying model has recently been published by this author [3] that introduces a new universal viscoelastic model to describe a definitive relationship between constant strain rate, creep and stress relaxation analysis for viscoelastic polymeric compounds. All three phases of the creep curve including primary, secondary and tertiary creep have been well represented using this model. Prior to the introduction of this new model several authors had attempted to describe two or more of these viscoelastic concepts in one unifying formulation [4, 5]. However, most of the effort over the years has been to simulate uniaxial creep [6–7], stress relaxation [4] or constant strain rate data [8–11] separately. This new formulation approach also offers a reasonably simple process in which to shift from a constant strain rate configuration to a creep calculation or stress relaxation configuration without changing formulation considerations or without stress or strain discontinuities.

While this new model has successfully characterized these three primary viscoelastic properties, the potential of a common failure criterion between these three properties has not yet been addressed in detail. This study will show that the application of an apparent common concept for failure between constant strain rate, creep and stress relaxation evaluations has been found to have some surprisingly different manifestations in practice. Since stress relaxation measurements have been addressed in some detail previously [3], this study will focus predominantly on a comparison between constant strain rate measurements and creep evaluations regarding failure criteria. For reference, this new universal viscoelastic model [3] will be briefly reviewed before introducing some new concepts to better understand creep failure phenomena.

1.1. Brief review of the universal viscoelastic model relating constant strain rate, creep and stress relaxation measurements

The basic universal viscoelastic model recently published elsewhere [3] begins with the most general equation to fit a stress strain curve which can be written as

$$\frac{\sigma}{\sigma_y} = K\varepsilon + A_2(K\varepsilon)^2 + A_3(K\varepsilon)^3 + A_4(K\varepsilon)^4 \cdots + A_n(K\varepsilon)^n \qquad (1)$$

where

$E =$ Elastic modulus (psi)
$\sigma_y =$ Engineering yield stress (psi)
$K = \frac{E}{\sigma_y} \approx$ Constant for a series of strain rates for the same polymer formulation
$A_2, A_3, \ldots A_{\text{i}} =$ Variable constants for a series of strain rates for the same polymer formulation

In this study only the first three constants in Equation 1 have been addressed as:

$$\frac{\sigma}{\sigma_y} = K\varepsilon + A_2(K\varepsilon)^2 + A_3(K\varepsilon)^3 \qquad (2)$$

According to Brown [9, 13, 14] and several other authors [10, 15] $K = \frac{E}{\sigma_y}$ is normally a constant for a given polymer formulation that typically ranges from 40–60. Also note that the stress, σ, from Equation 2 reduces to

$$\sigma \to E\varepsilon \quad \text{as} \quad \varepsilon \to 0 \qquad (3)$$

The strain, ε, in Equation 1 should be positive in a tension mode and negative for the compression mode. The analysis in this study assumed the strain, ε, to be positive and in a tension mode. The required modifications for the compression condition will be left to the reader.

The two other conditions required to evaluate the constants A_2 and A_3 in Equation 2 would include the following:

By definition: $\sigma = \sigma_y$ when $\varepsilon = \varepsilon_y$

Second condition: $d\sigma/d\varepsilon = 0$ at $\sigma = \sigma_y$ when $\varepsilon = \varepsilon_y$

Using these conditions it can be shown that if $K\varepsilon_y \le 3$

$$A_2 = \frac{(3 - 2K\varepsilon_y)}{K^2\varepsilon_y^2} \qquad (4)$$

$$A_3 = \frac{(K\varepsilon_y - 2)}{K^3\varepsilon_y^3} \qquad (5)$$

Thus if $\frac{d(\sigma/\sigma_y)}{d\varepsilon} = 0$ and if $K\varepsilon_y \le 3$, then the two extrema at $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$ can be found to yield a maximum at

$$\sigma_1 = \sigma_y \quad \text{at} \quad \varepsilon_1 = \varepsilon_y \qquad (6)$$

and a minimum at

$$\sigma_2 = \sigma_y\left(\frac{K^2\varepsilon_y^2(4K\varepsilon_y - 9)}{27(K\varepsilon_y - 2)^2}\right) \quad \text{at} \quad \varepsilon_2 = \varepsilon_y \times \left(\frac{K\varepsilon_y}{(3K\varepsilon_y - 6)}\right) \qquad (7)$$

The relationship between yield stress, σ_y, and time to yield, t_y, can be addressed using the following simple relationship currently included in ASTM D2837 -98a (Standard Test Method for Obtaining Hydrostatic Design Basis for Thermoplastic Pipe Materials):

$$\sigma_y = \frac{\beta}{t_y^n} \qquad (8)$$

Where $\sigma_y =$ the engineering yield stress, $t_y =$ time to yield, $n =$ Efficiency of Yield Energy Dissipation and $\beta =$ Constant. This relationship has also been used by Reinhart [12] to predict long term failure stress (which is normally close to the stress evaluated from the stress relaxation of the yield stress) as a function of time.

The calculated values of strain, ε, can also be evaluated on a time scale by noting that the time, t, to reach a given strain, ε, can be evaluated from the characteristic strain rate, $\dot{\varepsilon}_i$, as:

$$t = \frac{\varepsilon}{\dot{\varepsilon}_i} \qquad (9)$$

Also note that the yield strain, ε_y, and the time to yield, t_y, are also related by a characteristic strain rate, $\dot{\varepsilon}_i$, as:

$$t_y = \frac{\varepsilon_y}{\dot{\varepsilon}_i} \qquad (10)$$

Preliminary experimental measurements by this author as well as others in the literature [4, 5] have found that the strain at yield, ε_y, generally has been found to be a linear function at low strain rates, $\dot{\varepsilon}_i$, for constant strain rate measurements as:

$$\varepsilon_y = \varepsilon_\infty + \alpha\dot{\varepsilon}_i \qquad (11)$$

where, ε_∞, is the limiting strain to yield when the strain rate approaches an infinitely small value ($\dot{\varepsilon}_i \to 0$) and α is simply a small proportionality constant.

Brinson and DasGupta [4] point out that Crochet [16] predicted theoretically that the yield strain should decrease with an increase in strain rate. As indicated previously [3] this author has found that α is indeed negative for polyethylene. However, Malpass [5] and this author have found that for most ABS materials the strain to yield increases as the strain rate increases which would make α positive. In addition, Brinson and DasGupta [4] also found out experimentally that the yield strain increased with an increase in strain rate for polycarbonate.

However, it should be pointed out that the linear function described by Equation 11 appears to be a good approximation of the more detailed model at low strain rates. For the full range of strain rates and particularly for large strain rates then Equation 11 is better described by the following equation;

$$\varepsilon_y = \varepsilon_\infty + \varepsilon_0(1 - e^{-\gamma\dot{\varepsilon}_i}) \qquad (12)$$

This equation has the following limits

$$\varepsilon_y \rightarrow \varepsilon_\infty \quad \text{as} \quad \dot{\varepsilon}_i \rightarrow 0 \text{ (very long times)}$$

$$\varepsilon_y \rightarrow \varepsilon_\infty + \varepsilon_0 \quad \text{as} \quad \dot{\varepsilon}_i \rightarrow \infty \text{ (very short times)}$$

In addition, Equation 12 can also be simplified using a MacLaurin series of the exponential term to give

$$\varepsilon_y = \varepsilon_\infty + \varepsilon_0\left(\gamma\dot{\varepsilon}_i - \frac{\gamma^2\dot{\varepsilon}_i^2}{2} + \cdots\right) \qquad (13)$$

When $\dot{\varepsilon}_i \ll 1$ then Equation 13 reduces to

$$\varepsilon_y = \varepsilon_\infty + \varepsilon_0\gamma\dot{\varepsilon}_i \qquad (14)$$

Notice that Equation 14 is exactly the same as Equation 11 if

$$\alpha = \varepsilon_0\gamma \qquad (15)$$

In this study an ABS polymer will be used as an example of a viscoelastic material to illustrate the capabilities of the models presented. Based approximately on unpublished constant strain rate measurements made by this author the ABS material to be used as an example in this study will utilize the following constants ($\gamma = 50$ min., $\varepsilon_0 = 0.0044$ and $\varepsilon_\infty = 0.04$).

Substituting Equation 10 into Equation 8 then gives

$$\sigma_y = \beta\left(\frac{\dot{\varepsilon}_i}{\varepsilon_y}\right)^n \qquad (16)$$

Equation (16) can then be substituted into Equation 2 to give

$$\sigma = \beta\left(\frac{\dot{\varepsilon}_i}{\varepsilon_y}\right)^n \left[K\varepsilon + A_2(K\varepsilon)^2 + A_3(K\varepsilon)^3\right] \qquad (17)$$

While the yield strain, ε_y, is best described over the full range of strain rates by Equation 12, it is often convenient to use Equation 11 to simulate the yield strain, ε_y, at very low strain rates to give a simplified form of both Equations 16 and 17 as

$$\sigma_y = \beta\left(\frac{\dot{\varepsilon}_i}{\varepsilon_{\infty+\alpha\dot{\varepsilon}_i}}\right)^n \qquad (18)$$

$$\sigma = \beta\left(\frac{\dot{\varepsilon}_i}{\varepsilon_{\infty+\alpha\dot{\varepsilon}_i}}\right)^n \left[K\varepsilon + A_2(K\varepsilon)^2 + A_3(K\varepsilon)^3\right] \qquad (19)$$

Based on Equations 16–19 it is apparent that any tensile stress, σ, associated with a specific strain value, ε, including the yield strength, σ_y, will increase with an increase in the strain rate, $\dot{\varepsilon}_i$. However the strain to yield, ε_y, based on either Equation 11 or Equation 12 is only mildly sensitive to strain rate and is allowed to either increase or decrease slightly with an increase in the strain rate, $\dot{\varepsilon}_i$.

It is also interesting to address the case that exists at long times, t, or using Equation 19 at very low elongation rates, $\dot{\varepsilon}_i$. For this case note that the yield stress, ε_y, approaches a limiting value, ε_∞:

$$\varepsilon_y = \varepsilon_\infty + \alpha\dot{\varepsilon}_i \rightarrow \varepsilon_\infty \quad \text{as} \quad \dot{\varepsilon}_\mathrm{i} \rightarrow 0$$

For this case the constants A_2 and A_3 also approach the following values

$$A_2' = \frac{(3 - 2K\varepsilon_\infty)}{K^2\varepsilon_\infty^2} \qquad (20)$$

$$A_3' = \frac{(K\varepsilon_\infty - 2)}{K^3\varepsilon_\infty^3} \qquad (21)$$

and Equation 19 reduces to

$$\sigma = \beta\left(\frac{\dot{\varepsilon}_i}{\varepsilon_\infty}\right)^n \left[K\varepsilon + A_2'(K\varepsilon)^2 + A_3'(K\varepsilon)^3\right] \qquad (22)$$

Combining Equations 9 and 22 gives

$$\sigma = \beta\left(\frac{\varepsilon}{\varepsilon_\infty}\right)^n \left(\frac{1}{t^n}\right)\left[K\varepsilon + A_2'(K\varepsilon)^2 + A_3'(K\varepsilon)^3\right] \qquad (23)$$

Again it should be noted that Equations 22 and 23 apply only to the condition where the yield strain, ε_y, approaches its limiting value of ε_∞ as a result of the strain rate, $\dot{\varepsilon}_i$, approaching zero (0). Modification of Equation 23 can also be rearranged for creep analysis in the following form:

$$t = \left(\frac{\varepsilon}{\varepsilon_\infty}\right)\left(\frac{\beta}{\sigma}\right)^{1/n} \left[K\varepsilon + A_2'(K\varepsilon)^2 + A_3'(K\varepsilon)^3\right]^{1/n} \qquad (24)$$

As was indicated in a previous publication [3], Equations 22, 23 and 24 can be extremely helpful when trying to address either creep or stress relaxation at very low strain rates, $\dot{\varepsilon}_i$, or at very long times, t.

In general, Equations 1–19 can be used to describe a complete series of uniaxial constant strain rate curves for a given polymer formulation and/or processing condition as indicated for an example ABS type material in Fig. 1. These stress vs strain curves have been calculated primarily using Equation 17 from this new universal viscoelastic model for a series of strain rates from .002 in/min to 20 in/min. The yield stress vs yield strain curve also indicated in Fig. 1 was calculated primarily from Equations 16 and 12 respectfully. For reference, all of the constant strain rate curves in Fig. 1 were generated using Equation 17 with the following typical parameters for an ABS type polymeric material $K = 58$, $\gamma = 50$ min., $\varepsilon_0 = 0.0044$ and $\varepsilon_\infty = 0.04.$, $\beta = 4990$ psi and $n = .21$. Again these constants represent typical values for an ABS material as obtained from unpublished constant strain rate data generated by

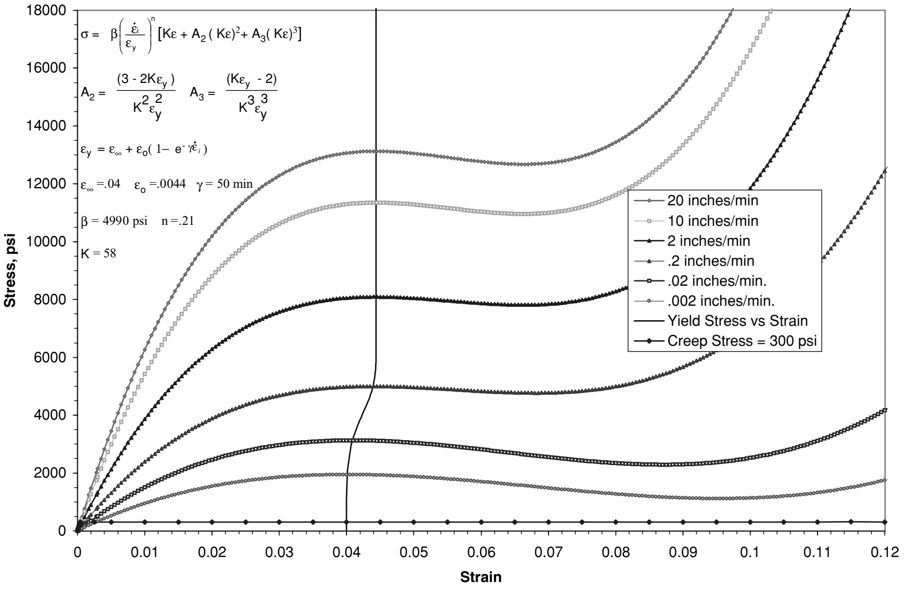

Figure 1 Calculated engineering stress vs strain at various strain rates for a simulated ABS material using the universal viscoelastic model with indications of creep at 300 psi.

this author. As described in a previous publication [3], a creep curve at a constant stress such as 300 psi as indicated in Fig. 1 can be developed from an identification of the strain at a series of points at the same stress level but from a series of constant strain rate curves.

The stress vs strain curves in Fig. 1 have been converted to stress vs time curves in Fig. 2 using Equation 9. The values for the yield stress vs time to yield in Fig. 2 have been obtained by applying Equation 8 or by applying Equation 10 to the strain to yield values shown in Fig. 1. For reference, the yield stress vs time to yield curve in Fig. 2 was again generated using the following typical parameters for an ABS type polymeric material with $\beta = 4990$ and $n = .21$. Again notice the locus of points that would make up the creep curve at 300 psi as indicated in Fig. 2. The generation of a creep curve from these data points will be discussed in more detail in the next section.

1.2. Creep curve generation using the new viscoelastic model

Creep is defined as the time dependent increase in strain of a viscous or viscoelastic material under sustained and constant stress. As indicated in Fig. 1 a creep curve can be developed from an identification of the strain at a series of points at the same stress level from a series of constant strain rate curves. However, initially the increase in stress to the level from which the creep curve can be initiated must be simulated. Typically the simplest simulation approach can be achieved from a constant strain rate process that can be used to arrive at the desired level of unchanging stress, σ_C, from which the creep process can begin. Equation 17 can then be used to generate this stress, σ, versus strain, ε, curve at a specific strain rate, $\dot{\varepsilon}_i$, until the desired level of stress, σ_C, is achieved from which a creep process can be initiated.

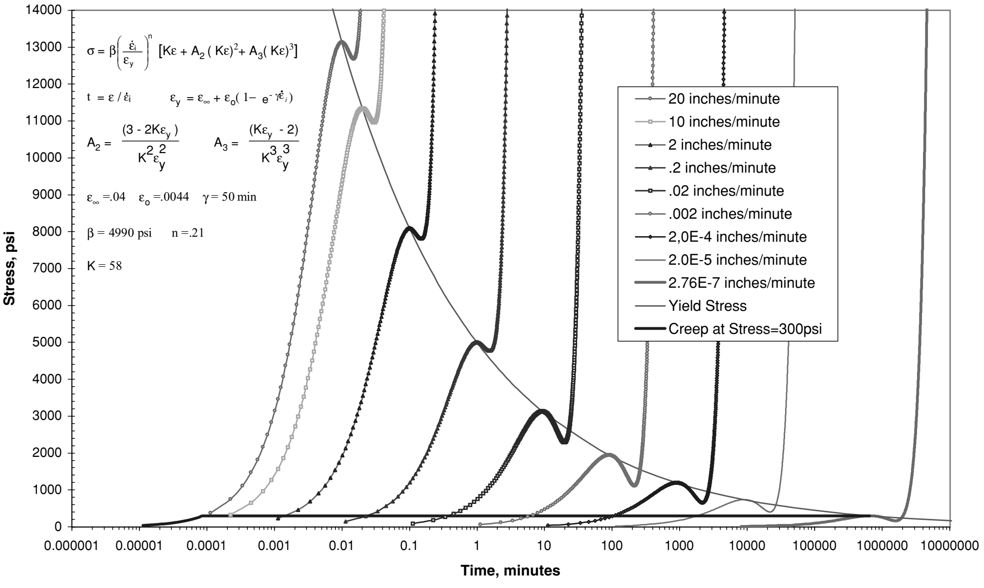

Figure 2 Calculated stress vs time at different strain rates using the universal viscoelastic model with indications of creep at 300 psi and yield stress relaxation.

Once the desired stress level, σ_C, has been reached using a constant strain rate approach, then Equation 19 must be solved for the strain to give a specific stress, σ_C, as the strain rate, $\dot{\varepsilon}_i$, is continued to be decreased to get to longer creep strains, ε_C, which can then be converted to creep times, t_C. Also note that after a designated very low effective strain rate, $\dot{\varepsilon}_i$, is reached the creep time, t_C, accumulated for a creep strain, ε_C, can be calculated directly from Equation 24. The locus of points involving calculated values of creep strain, ε_C, and the associated creep times, t_C, then constitutes the creep curve as indicated in Fig. 3.

There are four potential ways then to calculate creep strain, ε_C, using Equation 17 or 19 as a function of strain rate, $\dot{\varepsilon}_i$, to longer creep times, t_C. These four options include:

1. Solve Equation 17 or 19 as a cubic equation to calculate the appropriate creep strain, ε_C, at decreasing levels of strain rate, $\dot{\varepsilon}_i$, but at the desired creep stress level, σ_C. The creep time, t_C, accumulated for a specific creep strain, ε_C, at a specific strain rate, $\dot{\varepsilon}_i$, can then be calculated directly from Equation 9.
2. Solve Equation 17 or 19 using a numerical method such as the Newton-Raphson method to calculate the appropriate creep strain, ε_C, at decreasing levels of strain rate, $\dot{\varepsilon}_i$, but at the desired creep stress level, σ_C. The creep time, t_C, accumulated for a specific creep strain, ε_C, at a specific strain rate, $\dot{\varepsilon}_i$, can then be calculated directly from Equation 9.
3. Solve Equation 17 or 19 as a constant strain rate evaluation for each strain rate, $\dot{\varepsilon}_i$, and then solving the Equation 17 or 19 by trial and error for the creep strain, ε_C, that yields the desired creep stress, σ_C. The creep time, t_C, accumulated for a specific creep strain, ε_C, at each specific strain rate, $\dot{\varepsilon}_i$, can then be calculated directly from Equation 9.
4. Assuming the controlling strain rates, $\dot{\varepsilon}_i$, are very small after the constant level of creep stress, σ_C, is achieved and assuming the relative insensitivity of the values of A_2' and A_3' to strain rate at that point, then the creep curve can be calculated by close approximation directly using Equation 24. By setting the creep stress, σ_C, to a constant value then the creep time, t_C, can be calculated as a function of creep strain, ε_C, using Equation 24.

While method 3 appears to be very time consuming it can actually be evaluated relatively fast using a spreadsheet software such as MS Excel. This approach was also found to be particularly useful as the yield condition for creep or the inception of tertiary creep was approached and exceeded. If Option 3 is used, Equation 17 or 19 is first applied at a constant strain rate to increase the stress and associated strain until the desired stress level has been achieved. After the desired stress level has been reached, the successive strain values for the creep process can be developed by identifying the appropriate strain on successive stress strain curves that corresponds to the desired level of stress being evaluated. At this point each constant strain rate curve can be used to generate only one strain level at a given stress level on the creep curve as indicated in Figs 1 and 2. Since each constant strain rate curve can be described

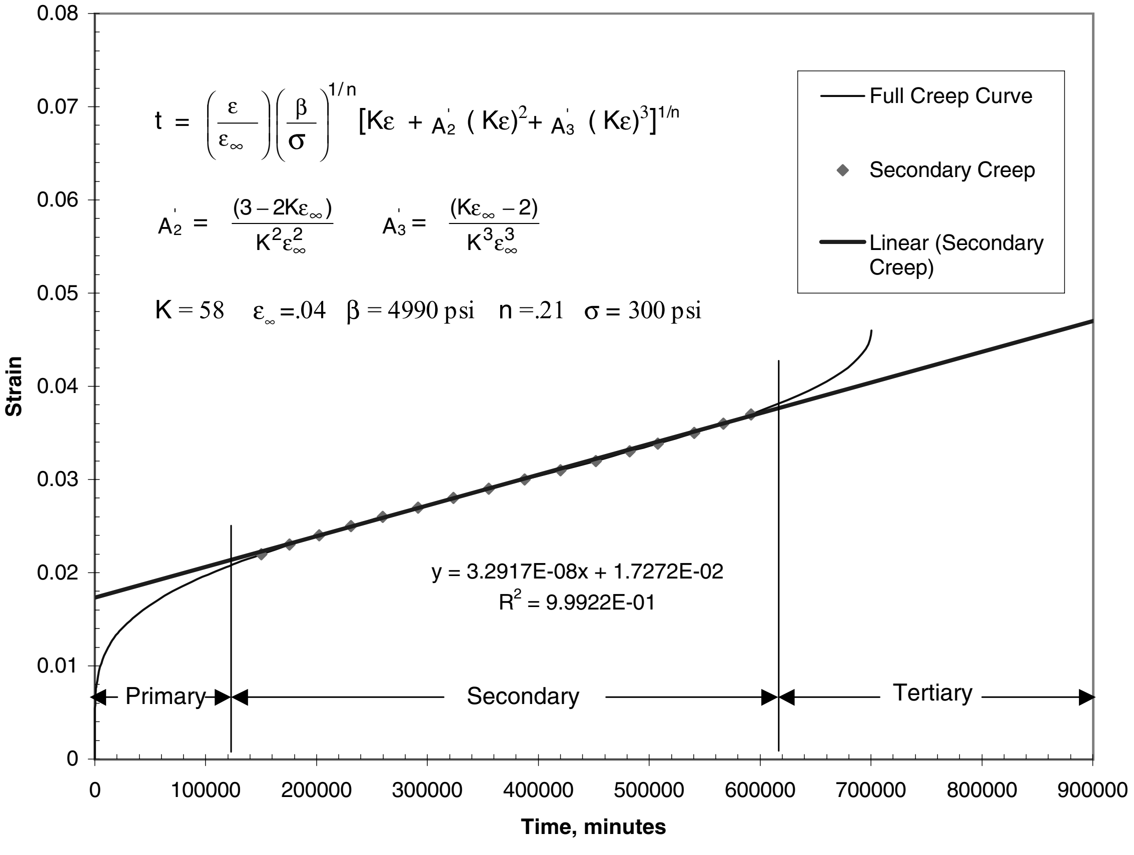

Figure 3 Calculated constant strain rate until the desired stress (300 psi) was reached followed by creep strain vs time and showing all three phases of creep.

by Equation 17 or 19, then these equations can be used to calculate the strain, ε_C, at the desired stress, σ_C and at the characteristic strain rate of, $\dot{\varepsilon}_i$, being addressed. For a creep process both the yield strain, ε_y, and the yield strength, σ_y, are functions of only the strain rate, $\dot{\varepsilon}_i$, as indicated in Equations 12 and 16. Also the time, t_C, for a specific strain, ε_C, at a specific strain rate, $\dot{\varepsilon}_i$, can be calculated directly from Equation 9. Therefore at a constant stress level, σ_C, the creep curve for a series of strain levels, ε_C, and their associated times, t_C, can be calculated from a series of constant strain rate stress strain curves. The locus of these calculated points then constitutes all three phases of the creep curve including primary, secondary and tertiary creep as indicated in Fig. 3.

According to Thorkildsen [17] Primary Creep includes all the initial changes in deformation prior to Secondary Creep. The first region of creep after Primary Creep that shows a linear increase in strain with time is called Secondary Creep. Tertiary Creep is the final stage of creep and it will be shown that this stage of creep can be correlated with the yield point from constant strain rate measurements.

Using the formulation concepts to calculate creep as discussed in this paper, the initial phase of a creep test begins with the constant strain rate component followed by the more typical creep process as indicated in Fig. 3. Of particular interest is the observation that the three different phases of the creep curve in Fig. 3 plot as straight lines when plotted on a log-log scale as indicated in Fig. 4.

If the simplifying assumptions of Option 4 are acceptable after some limiting strain rate, then use of Equation 24 allows the simplest approach to generate all three phases of the creep curve including primary, secondary and tertiary creep as indicated in Fig. 4. In addition, the creep results in Fig. 4 can be described over a much larger time scale in a very convenient fashion using Option 4. It is interesting to note in Fig. 4 that both Option 3 and Option 4 give the same creep curve up until the inception of tertiary creep. At this point there is a jump in the data using Option 3 but the results for Option 4 yield quite a different but continuous curve near the condition as defined by yield point failure criterion. An elucidation of the apparent conflict between Options 3 and 4 near creep failure as indicated in Fig. 4 will be addressed in the next sections of this paper.

1.3. Manifestations of yield stress failure criteria on creep using the universal viscoelastic model

The first maximum stress (where $\mathrm{d}\sigma/\mathrm{d}\varepsilon = 0$) on each of the constant strain rate curves in Fig. 1 has often been defined as the yield stress and this yield point can be identified by a specific stress at a specific strain. The yield point for constant strain rate data is often considered at least one condition for failure in a polymer or polymer compound. This failure condition can then be applied to the creep process to see how it might be manifested using the model described in this study.

The first manifestation of the yield stress failure criterion as applied to creep can be elucidated in Fig. 2. The three different types of curves indicated in Fig. 2 include the constant strain rate curves involving stress vs time, the creep curve at 300 psi and the locus of points describing the stress relaxation of the yield stress as a function of the time to yield. The failure condition for creep at 300 psi would then appear to occur at the intersection of the yield stress relaxation curve and the creep curve at 300 psi yield stress. This failure criterion involves primarily stress as related to the yield stress.

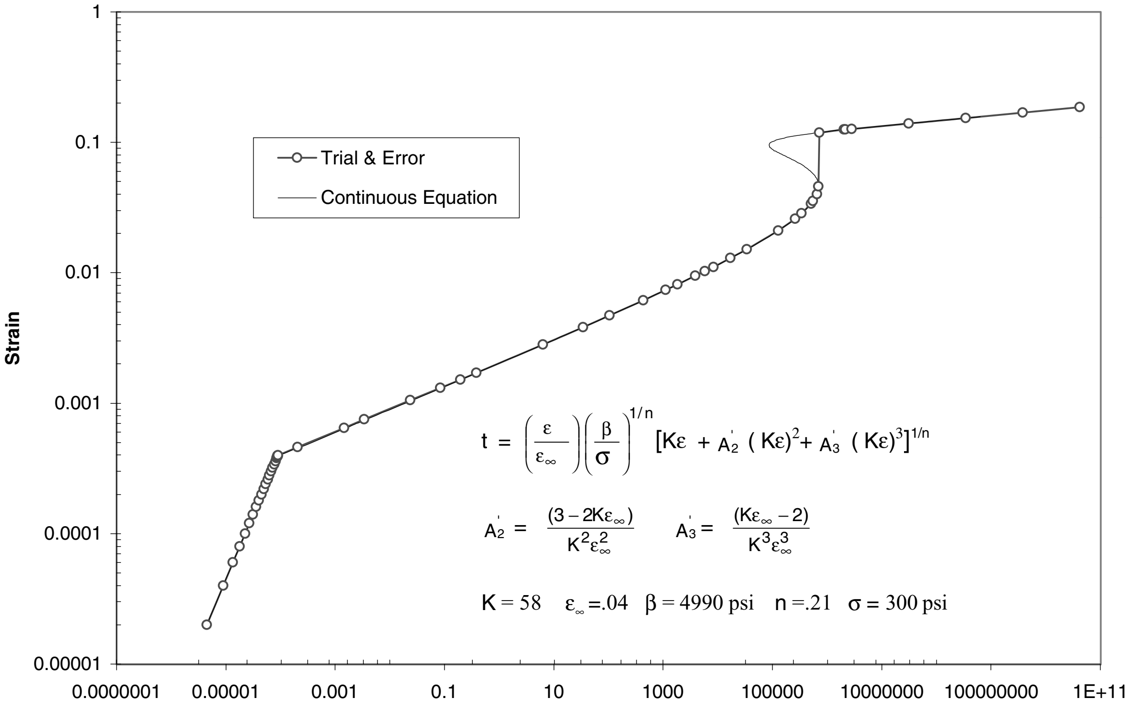

Figure 4 Calculated plot of long term creep at 300 psi stress showing the linear character of strain at both short and long times on a log-log scale.

The creep time anticipated at failure while maintaining a constant creep stress would then be obtained from the time to yield.

At very long times or very low strain rates the yield strain approaches the limiting yield strain of $\varepsilon_\infty = .04$ for the simulated ABS material evaluated in this study. Therefore, a second way to describe the yield point failure criterion for creep at 300 psi would be to identify the point described by the creep time required to reach the limiting strain to yield ($\varepsilon_\infty = .04$) as indicated in Fig. 3. The creep strain of $\varepsilon_\infty = .04$ occurs at the very end of the straight line portion of secondary creep as indicated in Fig. 3. In other words the strain at $\varepsilon_\infty = .04$ is also the strain at which tertiary creep appears to be initiated. Thus, if the strain to yield is assumed to be nearly a constant at very low cross head speeds or at long creep times, then the strain to yield at long creep times, ε_∞, is apparently the same as the strain at the inception of tertiary creep, $\varepsilon_{\mathrm{itc}}$. As indicated earlier, the yield point for constant strain rate data is considered in some circles to be at least one condition for failure in a polymer or polymer compound. Therefore, the creep time associated with the strain at the inception of tertiary creep, $\varepsilon_{\mathrm{itc}}$, would be expected to be consistent with the criterion described by yield failure at the limiting yield strain, ε_∞, for constant strain rate data.

1.4. Apparent failure criterion for creep that is slightly different from yield point criterion

As discussed earlier, all three phases of the creep curve including primary, secondary and tertiary creep are indicated in Fig. 4. Up to the inception of tertiary creep the results in Fig. 4 show that the same creep curve is obtained for both Options 3 and 4. However, slightly after the condition defined by yield point failure criterion there is an apparent jump in the data using Option 3 while a continuous curve is obtained for Option 4. An elucidation of the apparent conflict between Options 3 and 4 is much easier to understand if we look closer at the realistic nature of these two methods for evaluating creep. For option 4 the strain determines the calculated creep time as indicated in Equation 24. As the strain increases from the inception of tertiary creep it can be seen in Fig. 4 that eventually a strain is reached where Option 4 calculations actually would require a step back in time to go to the next differential element of strain. Since going back in time is currently impossible, only the jump in strain calculated by option 3 for the next element of time would be realistic. However, for Option 3 a huge jump in strain is required to go to the next element of time. Since this condition is already slightly beyond the inception of tertiary creep, then if failure did not occur at the inception of tertiary creep it would almost surely be expected to fail at this condition.

The jump-in-strain creep failure criterion is sensitive to several variables and one such variable is the constant stress level as indicated in Fig. 5. The location of the strain at which this jump in strain is required at a given stress level has been designated as the strain at "Critical Creep", ε_{CC}, and can be identified by taking the derivative of Equation 24

$$t = \left(\frac{\varepsilon}{\varepsilon_\infty}\right)\left(\frac{\beta}{\sigma}\right)^{1/n}\left[K\varepsilon + A_2'(K\varepsilon)^2 + A_3'(K\varepsilon)^3\right]^{1/n} \tag{24}$$

The derivative of Equation 24 gives

$$\frac{\mathrm{d}t}{\mathrm{d}\varepsilon} = \frac{t}{\varepsilon}\left[1 + \frac{1}{n}\left(\frac{1 + 2A_2'(K\varepsilon) + 3A_3'(K\varepsilon)^2}{1 + A_2'(K\varepsilon) + A_3'(K\varepsilon)^2}\right)\right] \tag{25}$$

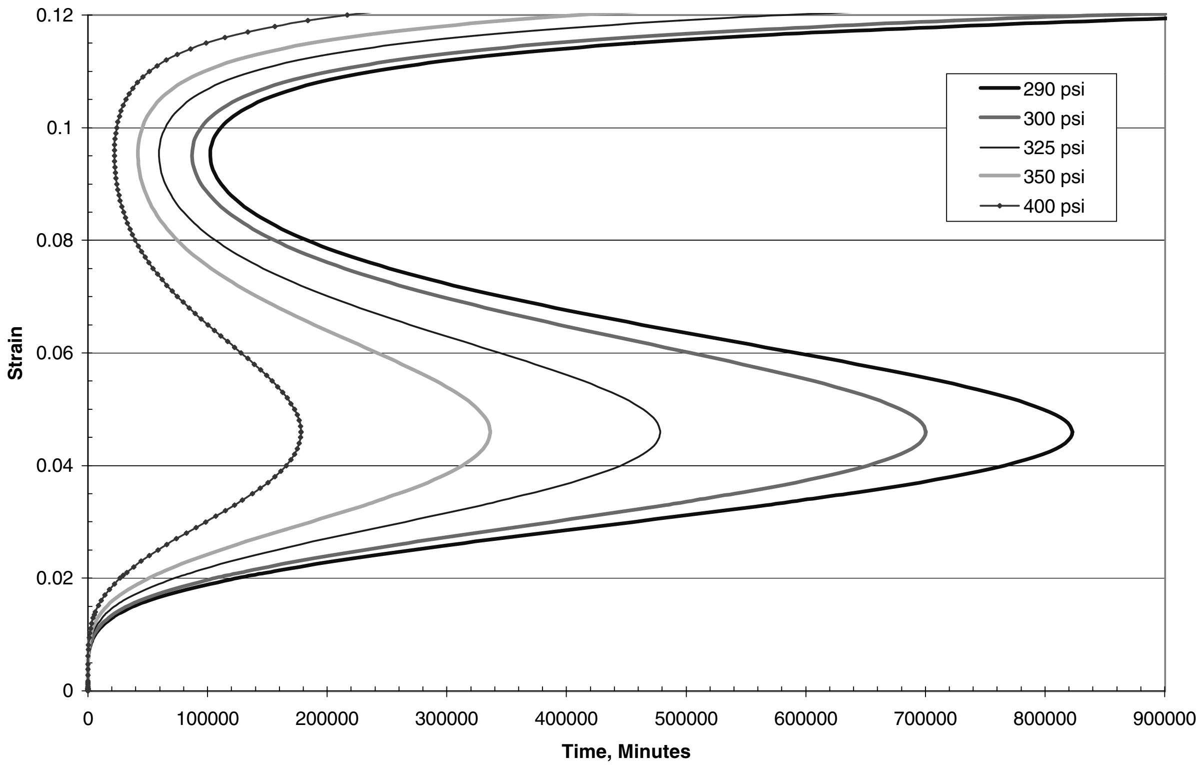

Figure 5 Creep strain vs time at different levels of creep stress for example ABS Material ($n = .21$).

Thus when $\frac{\mathrm{d}t}{\mathrm{d}\varepsilon} = 0$ then Equation 25 reduces to

$$n = -\left(\frac{1 + 2A_2(K\varepsilon_{\mathrm{CC}}) + 3A_3(K\varepsilon_{\mathrm{CC}})^2}{1 + A_2(K\varepsilon_{\mathrm{CC}}) + A_3(K\varepsilon_{\mathrm{CC}})^2}\right) \tag{26}$$

Equation 26 can then be solved for the strain at "critical creep," $\varepsilon_{\mathrm{CC}}$, to give

$$\varepsilon_{\mathrm{CC}} = \left(\frac{-(n+2)A_2' \pm \sqrt{(n+2)^2 {A_2'}^2 - 4(n+1)(n+3)A_3'}}{2\,(n+3)\,A_3' K}\right) \tag{27}$$

The maximum or the minimum value of the strain at critical creep, $\varepsilon_{\mathrm{CC}}$, occurs when the square root component of Equation 27 is equal to zero (0) to give

$$n = -2 \pm 2\sqrt{\frac{K\varepsilon_\infty(K\varepsilon_\infty - 2)}{4\,K\varepsilon_\infty - 9}} \tag{28}$$

Also note that when $n = 0$, then Equation 27 yields "Critical Creep" at

$$\varepsilon_{\mathrm{CC}} = \varepsilon_\infty \tag{29}$$

It should also be noted that a second extra extrema strain, $\varepsilon_{\mathrm{EE}}$, is also achieved when $n = 0$ at

$$\varepsilon_{\mathrm{EE}} = \varepsilon_\infty\left(\frac{K\varepsilon_\infty}{3(K\varepsilon_\infty - 2)}\right) \tag{30}$$

It is also desireable to calculate the creep slope, $\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$. However, even though we do not have a direct relationship with strain as a function of time we do we do have the derivative, $\frac{\mathrm{d}t}{\mathrm{d}\varepsilon}$, described by Equation 25. Note that the reciprocal of Equation 25 then gives the desired derivative or slope as

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\varepsilon n}{t}\left(\frac{1 + A_2'(K\varepsilon) + A_3'(K\varepsilon)^2}{1 + n + (2+n)\,A_2'(K\varepsilon) + (3+n)\,A_3'(K\varepsilon)^2}\right) \tag{31}$$

The derivative described by Equation 31 has been plotted in Fig. 6 for creep at a constant stress of 300 psi. As indicated in Fig. 6, the instantaneous slope, $\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$, for creep at a constant stress of 300 psi has been generated by combining Equations 24 and 31 for the same ABS example material. This instantaneous slope in Fig. 6 is directly related to the slope of graphs in Figs 3 and 4. Note in Fig. 6 that the derivative, $\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$, or creep slope approaches a non-zero minimum but nearly constant value in the secondary creep region which appears to run from a strain of approximately $\varepsilon = .016$ to $\varepsilon = .04$. The slope in Fig. 6 also goes through two unstable conditions where the slope flips from one sign and then goes back to a slope of the opposite sign at a strains of approximately $\varepsilon_{\mathrm{CC}} = .046$ and $\varepsilon_{\mathrm{EE}} = .95$. The exact location of the strains at which these unstable slope changes occur can be obtained from Equation 27.

It is important to recognize that Equation 27 is independent of creep stress, σ_C. This means that the strain at critical creep, $\varepsilon_{\mathrm{CC}}$, will always occur at the same strain independent of creep stress as indicated in Fig. 5. Therefore the strain at critical creep, $\varepsilon_{\mathrm{CC}}$, evaluated

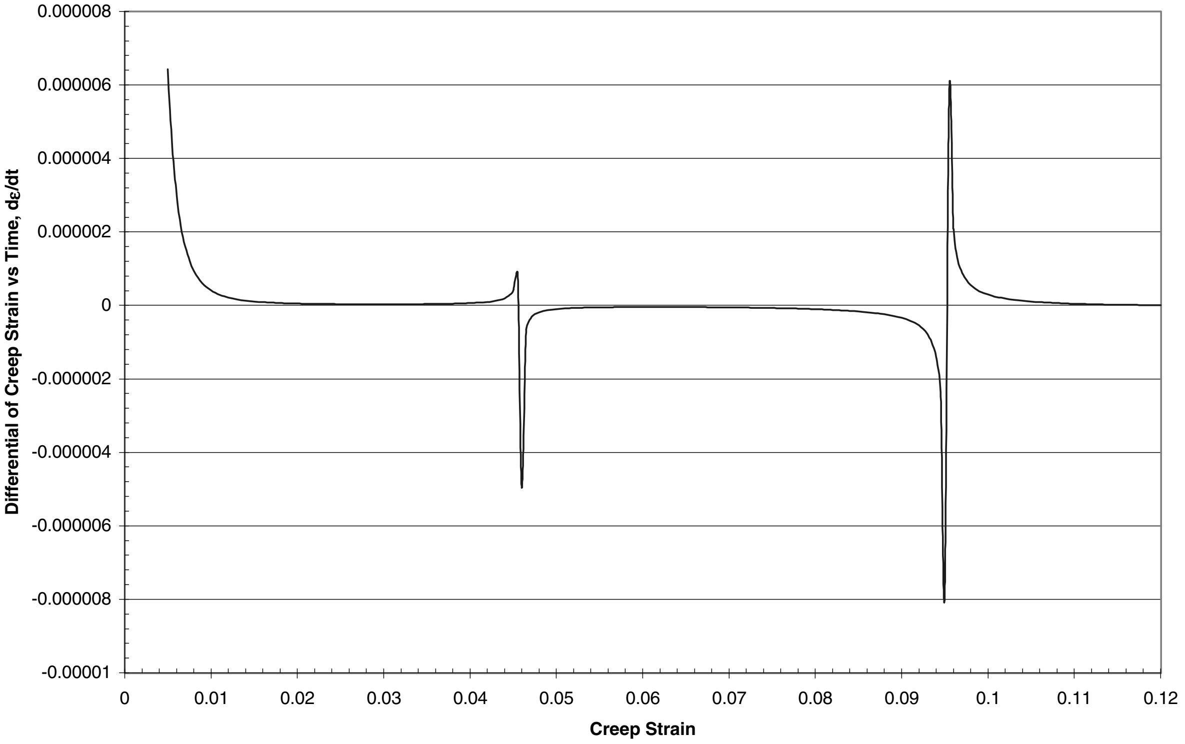

Figure 6 Differential (or slope) of creep strain vs time, dε/dt, vs creep strain (creep stress = 300 psi).

from Equation 27 is only dependent on the efficiency of yield energy dissipation, n, the ratio of modulus to yield stress, K, and the limiting value of the yield strain, ε_∞. If we again use the simulated ABS material as the example material with $n = .21$, $K = 58$ and $\varepsilon_\infty = .04$, then Equation 27 gives the "Critical Creep" strain, ε_{cc}, at

$$\varepsilon_{CC} = 0.04592$$

For this example the critical creep strain is only 15% greater than the yield strain criteria where the strain at the inception of tertiary creep would be $\varepsilon_{itc} = \varepsilon_\infty = .04$.

Using Equation 24 it is also easy to see that the ratio of the critical creep time, t_{CC}, or the time to reach the critical creep strain, ε_{CC}, at and the time to the inception of tertiary creep, t_{itc}, from the strain at the inception of tertiary creep, $\varepsilon_{itc} = \varepsilon_\infty$, would also be independent of the creep stress. However, as indicated in Equation 24 the time to failure for these two different techniques are not a necessarily direct ratios of their strains. Using the ABS example again the critical creep *time* can be shown to be only 7.42% greater than the time to the inception of tertiary creep independent of creep stress.

These results seem to explain at least two separate experimental observations that this author has witnessed regarding the way that creep failure occurs in ABS Materials. First, the failure condition in creep for ABS materials almost always occurs at a strain that is just slightly greater than the failure strain and failure time predicted from yield criterion. This result would be very consistent with the time to critical creep being just 7.5% greater than the time to reach the yield condition as predicted for the example ABS material. It should also be pointed out that the order of magnitude accuracy of the measurement of the time to creep failure is also often no better than 7.5%. Therefore, since time to critical creep and the time to creep failure from yield criterion can sometimes be very difficult to separate, it often appears that these two conditions may just simply be manifestations of the same failure phenomena.

The second experimental observation is that creep failure for ABS materials often involves a sudden but rapid increase in strain just prior to failure. The shape of the creep curves at different creep stresses as indicated in Figs 4 and 5 would appear to explain this phenomena. In order to avoid going back in time it would appear that a sudden strain would be required at critical creep to reach the next stable strain condition in the next element of time. It is easy to visualize how such a failure could be catastrophic—which in fact it is.

1.5. Influence of the efficiency of yield energy dissipation, *n*, on creep

In a recent review by this author [18] the power law constant, n, was found to be a dampening factor for the rate of dissipation of the available energy/volume relative to time in going from one strain rate curve to another at the yield condition. Consequently, the power law constant n was designated [18] the *Efficiency of Yield Energy Dissipation* with an effective range of $0 \le n \le 1$. Hernandez-Jimenez *et al.* [19] also recently reviewed several different models that have also tried to justify a theoretical development for the constant n. Most of these models referred back to the original model by Scott-Blair [20] who justified the constant n from a fractional derivative for a viscoelastic material. If the material being described can be considered to be viscoelastic, then Scott-Blair argued that the value of n must exist in the range from 0 (elastic solid) $< n <1$ (viscous liquid). Others have expanded on Scott-Blair's analysis as recently reviewed by Jimenez *et al.* [19].

The major advantage of the Scott-Blair analysis is that it does address a nice explanation of why the value of the "efficiency of yield energy dissipation," n, should only range from 0 to 1. The sensitive nature of the value of the "efficiency of yield energy dissipation," n, relative to the long range failure of materials is clearly seen in Fig. 7 for a constant stress of 300 psi. In this figure when the value of n was decreased 10% from .22 to .20 the corresponding critical creep failure time increased 257% from .73 years to 2.59 years. The influence of a much larger range of n is illustrated in Fig. 8 for a creep stress of 300 psi. As the value of n was increased from .2 to 1.0 in Fig. 8 the critical creep failure time decreased from 2.59 years to 22 minutes. The calculated results indicated in Figs 7 and 8 clearly show that if viscoelastic materials are intended to survive considerable application stress for very long times they need to have "efficiency of yield energy dissipation" values of $n < .4$ to be practical. This result is certainly consistent with experimental data. An explanation for this phenomena based on the Scott-Blair analysis would describe such a viscoelastic material as having a more elastic-solid like character than a viscous-liquid like character since n would be closer to 0 than to 1.

Also note that the expected creep failure strains are clearly indicated in Fig. 8 by the critical creep values. These results indicate that as n increases to values larger than $n > .4$ then the strain to critical creep also increases to values that can be significantly greater than the inception of tertiary creep. Thus the near equivalency of the critical creep criterion and the yield stress criterion is significantly more valid the lower the value of the efficiency of yield energy dissipation.

It is clear from Figs 5–8 that there should be two extrema solutions to Equations 26 and 27 for each level of n as indicated in Fig. 9. However, of these two extrema the value for critical creep, ε_{CC}, would be the strain of primary interest and this critical creep strain should be just slightly greater than the strain at the inception of tertiary creep. The second extra extrema strain, ε_{EE}, is not real since it only exists if it is possible to go back in time as indicated in Figs 7 and 8.

Also note in Fig. 9 that a maximum value of the efficiency of yield energy dissipation, n, is readily apparent. Since $K = 58$ and $\varepsilon_\infty = .04$ for the example ABS material, then $K\varepsilon_\infty = 2.32$ and the maximum efficiency of yield energy dissipation can be found to occur at of

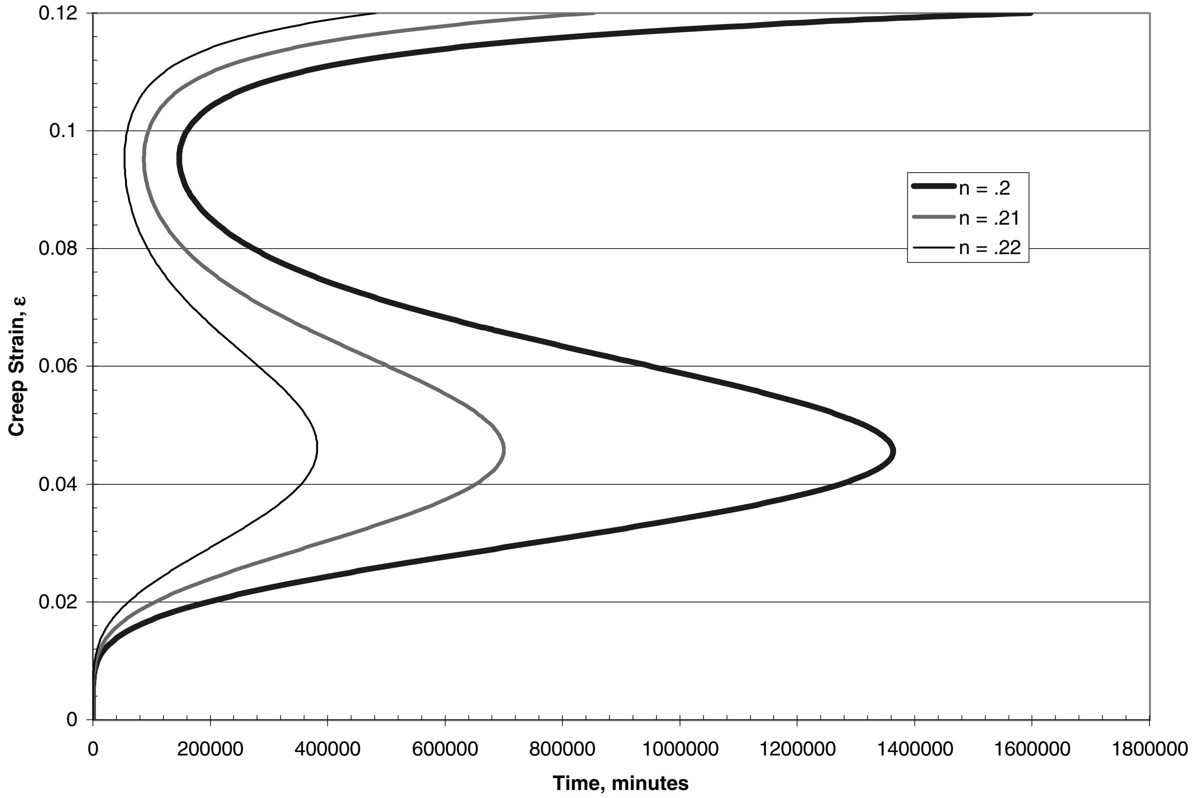

Figure 7 Creep strain vs time at three different levels of the efficiency of yield energy dissipation, n, for the example ABS material (stress = 300 psi).

$n = 1.257$ using Equation 28. At this maximum value of n, then a critical creep stain of $\varepsilon_{CC} = 0.07842$ would be obtained using Equation 27. While it is possible to calculate such a maximum as indicated in Fig. 9, it is not expected that such a material would be expected to have much elastic character based on a Scott-Blair type analysis and would therefore not be expected to have much practical interest as a viscoelastic structural

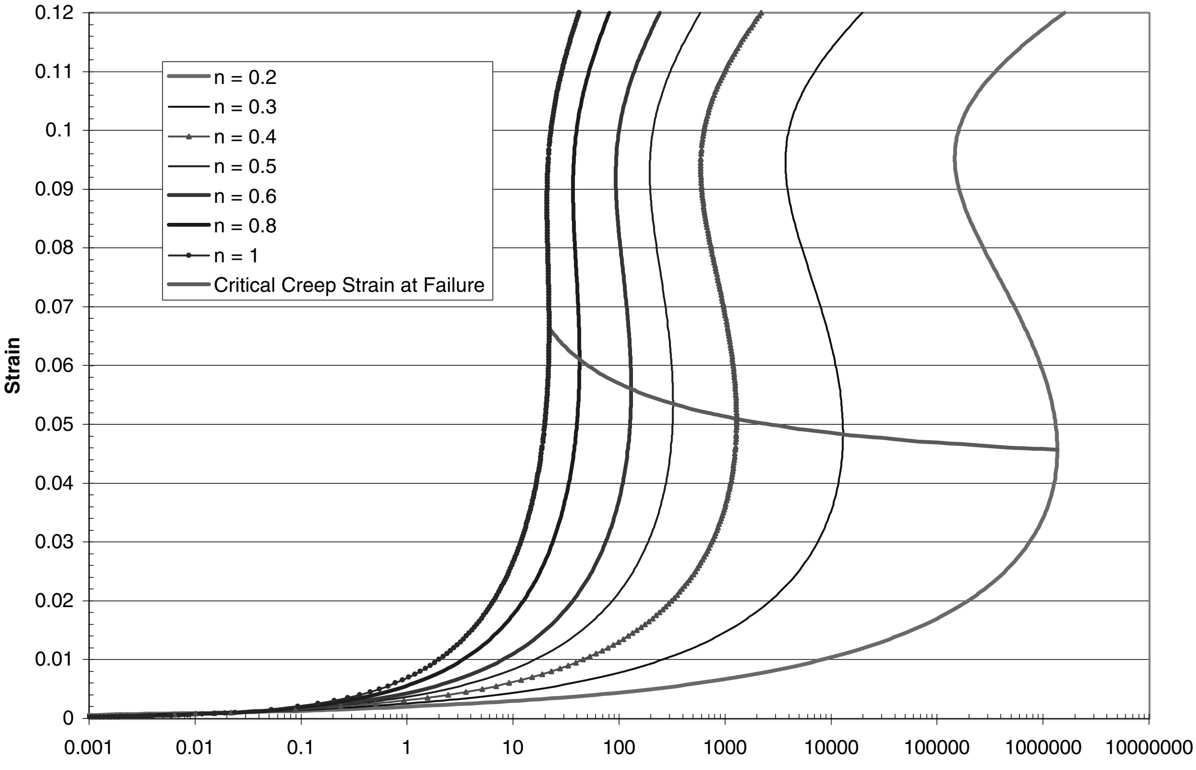

Figure 8 Creep strain vs time at levels covering the full range of the efficiency of yield energy dissipation, n, for the example ABS material (stress = 300 psi).

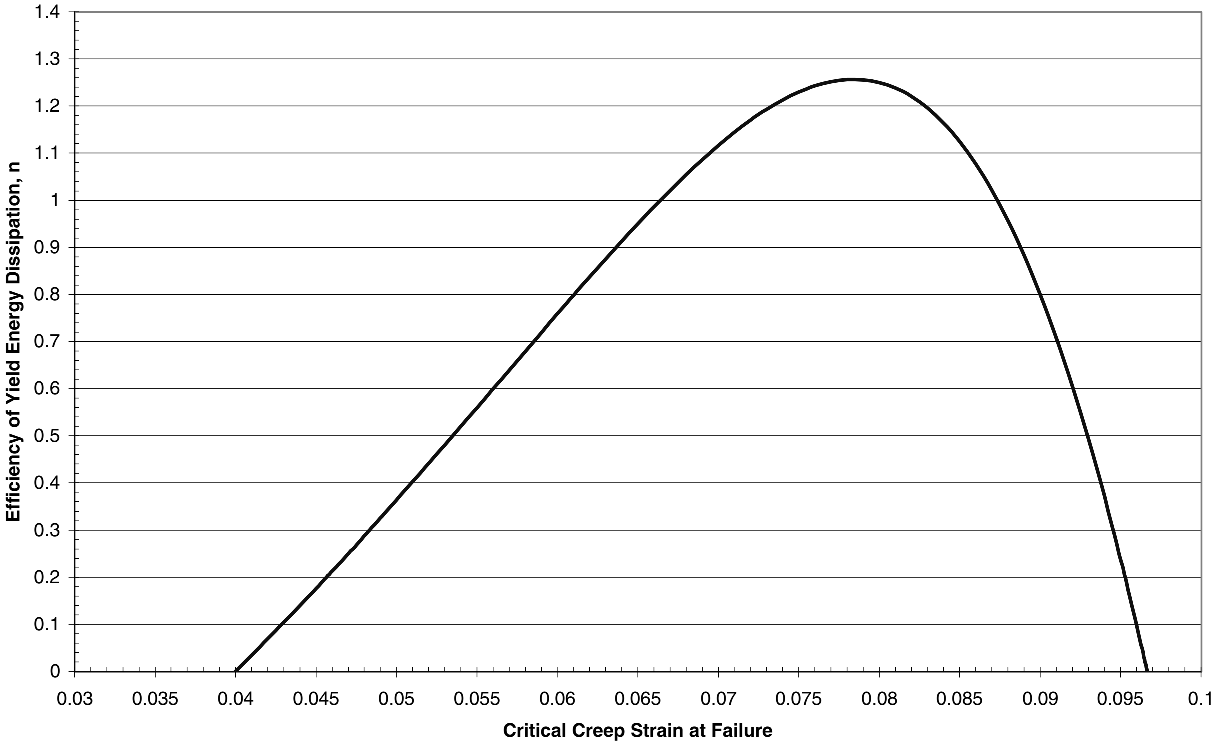

Figure 9 Efficiency of yield energy dissipation vs critical creep at failure for ABS example material.

material. While Equation 28 is only a function of the product $K\varepsilon_\infty$, it would not be reasonable to expect that a range of values for the efficiency of yield energy dissipation, n, could be achieved while retaining the same constant value for the product $K\varepsilon_\infty$.

While the maximum indicated in Fig. 9 is interesting, it is expected that material formulations will not be controllable enough in the near future to be able to generate such range of viscoelastic character using only formulation modifications for the same base material. It is also expected that significant experimental data will be needed to be able generate adequate control of the viscoelastic character of materials into the foreseeable future. Until proven otherwise it would be reasonable to assume that all the constants in the universal viscoelastic model described in this study will change at the same time but not necessarily in some predictable fashion. As a result, if one is interested in the long range failure of materials, then at least for the near future the viscoelastic constants needed for the universal viscoelastic model used in this study will need to be evaluated for each viscoelastic material formulation generated.

2. Conclusion

In a preceding publication this author introduced a new universal viscoelastic model to describe a definitive relationship between constant strain rate, creep and stress relaxation analysis for viscoelastic polymeric compounds. Since the failure criterion for creep for this model had not been addressed in detail in previous publications, this study focused predominantly on a comparison between constant strain rate measurements and creep evaluations regarding failure criteria.

The first manifestation of the yield stress failure criterion as applied to creep was elucidated at the intersection of the yield stress relaxation curve and the creep stress vs time curve. This stress failure criterion was applied from the yield stress and the time to yield to the creep time anticipated at failure while maintaining a constant creep stress.

At very long times or very low strain rates the strain at yield approaches the limiting yield strain, ε_∞. Therefore, a second way to describe the yield point failure criterion for creep would be to identify the point described by the creep time required to reach the limiting strain to yield, ε_∞ . The creep strain at ε_∞ occurs at the very end of the straight line portion of secondary creep that is also the strain at which tertiary creep appears to be initiated, $\varepsilon_{\mathrm{itc}} = \varepsilon_\infty$. Thus, if the strain to yield is assumed to be nearly a constant at very low cross head speeds or at long creep times, then the strain to yield at long creep times is apparently the same as the strain at the inception of tertiary creep. Therefore, the creep time associated with the strain at the inception of tertiary creep would be expected to be consistent with the failure criterion described by yield strain at the yield point for constant strain rate data.

Four potential ways to calculate creep strain as a function of time were described for the model used in this study. Three of these methods were iterative calculations and a fourth option was a simplified non-iterative version useful for evaluating long term creep. As the strain increases from the inception of tertiary creep it was found that eventually a critical creep strain, $\varepsilon_{\mathrm{CC}}$, is reached where the simplified continuous curve calculations actually would require a step back in time to go to the next differential element of strain. Since going back in time is currently impossible, only the huge jump in strain calculated by one of the trial and error approaches for the next element of time was found to be realistic. Since this critical creep strain, $\varepsilon_{\mathrm{CC}}$, is slightly larger than the inception of tertiary creep, then if failure did not occur at the inception of tertiary creep it would almost surely be expected to fail at this condition.

This study found that the derivative of creep time vs creep strain went through zero and back to a slope of the opposite sign at two strains—one of which is the critical creep strain, $\varepsilon_{\mathrm{CC}}$. It was found that the location of the critical creep strain, $\varepsilon_{\mathrm{CC}}$, was dependent only on the efficiency of yield energy dissipation, n, the ratio of modulus to yield stress, K, and the limiting value of the yield strain, ε_{∞}, and was essentially independent of creep stress.

For the ABS example in this study the critical creep strain was 15% greater than the yield strain criteria while the time to critical creep was only 7.42% greater than the time to the inception of tertiary creep. These results seem to explain at least two separate experimental observations regarding creep failure in ABS Materials. First, the failure condition in creep for ABS materials almost always occurs at a strain that is just slightly greater than the failure strain and failure time predicted from yield criterion. The second experimental observation is that creep failure for ABS materials often involves a sudden but rapid increase in strain just prior to failure. In order to avoid going back in time it would appear that a sudden large increase in strain would be required at critical creep to reach the next stable strain condition in the next element of time. It is easy to visualize how such a failure could be catastrophic—which in fact it is.

In a recent review by this author [18] the *Efficiency of Yield Energy Dissipation*, n, was found to have an effective range of 0 (elastic solid) $< n < 1$ (viscous liquid). In addition, if viscoelastic materials must survive considerable application stress for very long times they need to have "efficiency of yield energy dissipation" values of $n < .4$ to be practical and consistent with experimental data. An explanation for this phenomena based on the Scott-Blair analysis would describe such a viscoelastic material as having a more elastic-solid like character than a viscous-liquid like character since n would be closer to 0 than to 1.

This study also found that as efficiency of yield energy dissipation, n, increases to values larger than $n > .4$ then the strain to critical creep also increases to values that can be significantly greater than the inception of tertiary creep. Thus the near equivalency of the critical creep strain criterion and the yield strain criterion was found to be much more probable the lower the value of efficiency of yield energy dissipation such that $0 < n \ll .4$.

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Received 25 June
and accepted 25 November 2002